Complex Analysis Final Exam

November 22 2010

There are 5 questions of 10 marks each. Please look over the entire paper before attempting to answer as some questions may be easier than others.

1a. Let f be a function holomorphic on the open disc D. Prove that if $\arg(f)$ is constant then $|f|$ is constant. (5 marks)

1b. Prove that if u is a real valued harmonic function on D and u^2 is also harmonic then u is constant. (5 marks)

2a Prove that a linear fractional transformation that has only one fixed point is conjugate to a translation. That is, if $\gamma(z) = z$ for only one $z \in \mathbb{C}$ then there exist a linear fractional transformation α such that, for some $b \neq 0$, (3 marks)

$$\alpha\gamma\alpha^{-1} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

2b. Let $\mathfrak{H} = \{z | Im(z) > 0\}$ be the *upper half plane*. Prove that a linear fractional transformation γ maps $\mathfrak{H}$ to $\mathfrak{H}$ if and only if γ is a real 2×2 matrix of determinant 1 - that is, $\gamma \in SL_2(\mathbb{R})$. (7 marks)

3. Show that

$$\frac{1}{2\pi}\int_0^{2\pi} \cos^{2n}(t)dt = \frac{1 \cdot 2 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}$$

by integrating the function $\frac{1}{z}\left(z + \frac{1}{z}\right)^{2n}$ around the unit circle. (10 marks)

4a.Let p and q be polynomials such that $\deg(q) > 1 + \deg(p)$. Prove that the sum of the residues of the rational function $\frac{p}{q}$ taken over all its poles in $\mathbb{C}$ is 0. (6 marks)

4b. Evaluate

$$\int_C \frac{z dz}{(2z^3 - 1)(z + 2)}$$

where C is the unit circle taken counterclockwise. (4 marks)

5. Evaluate $\int_0^\infty \frac{\sin^2(x)}{x^2} dx$ by integrating the function $\frac{1 - e^{2iz}}{z^2}$ around the contour $\Gamma_{\epsilon,R}$ and letting $R \to \infty$ and $\epsilon \to 0$. $\Gamma_{\epsilon,R}$ is the contour which consists of two half circles, one of radius R oriented counterclockwise and one of radius ϵ, oriented clockwise, in the upper half plane, along with the lines along the real axis joining $-R$ to $-\epsilon$ and ϵ to R. (10 marks)